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L^p-Spaces for UHF Algebras

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We construct L^p-spaces for a UHF algebra A with a faithful product state φ. We show that the spaces obtained are isomorphic, as Banach spaces, to the Haagerup spaces $L^p(\pi_\varphi(A)'')$.

INTRODUCTION

The recent progress in the theory of noncommutative Dirichlet forms (see, for example, Davies and Lindsay, 1992; Goldstein and Lindsay, 1995) shows that L^p-techniques can be applied fruitfully to the theory of quantum dynamical semigroups. The use of properly defined noncommutative L^p-spaces also gives a very natural setting for hypercontractivity questions. We present a simple attempt at a construction of noncommutative L^p-spaces for a class of C^*-algebras. We consider a very simple situation, namely that of a UHF algebra A with a faithful product state φ. The reward is a very satisfactory result connecting the spaces with the Haagerup spaces for the von Neumann algebra generated by the image of A in the GNS representation with respect to the state φ. In many situations it will be possible to calculate the L^p-norms of A explicitly. The whole construction can be generalized easily, but the main ideas remain the same. We can find them, in a slightly different setting, in a recent paper of Majewski and Zegarlinski (1995). Consult Terp (1981) for the Haagerup theory and Trunov (1979) for the construction of L^p-spaces for semifinite algebras.

1. THE HAAGERUP SPACES

Let $\mathscr{A}$ be a von Neumann algebra acting in a Hilbert space H and ψ a faithful normal semifinite weight on $\mathscr{A}$. Denote by σ^ψ the modular auto-

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morphism group $\{\sigma_t^\psi\}_{t\in\mathbb{R}}$ for the pair $(\mathcal{A}, \psi)$. The crossed product $\overline{\mathcal{A}} = \mathcal{A} \rtimes_{\sigma^\psi} \mathbb{R}$ is the von Neumann algebra acting on $\overline{H} = L^2(\mathbb{R}, H)$ generated by operators $\pi(a)$, $a \in \mathcal{A}$, and $\lambda(s)$, $s \in \mathbb{R}$, defined by

$$(\pi(a)\xi)(t) = \sigma_{-t}^\psi(a)\xi(t) \qquad \text{for} \quad \xi \in \overline{H}, \quad t \in \mathbb{R}$$

$$(\lambda(s)\xi)(t) = \xi(t-s) \qquad \text{for} \quad \xi \in \overline{H}, \quad t \in \mathbb{R}$$

Let h be a positive self-adjoint operator on $\overline{H}$ such that $\lambda(t) = h^{it}$ for $t \in \mathbb{R}$, and let $\overline{\psi}$ be the dual weight on $\overline{\mathcal{A}}$. Put $\overline{\tau}(\cdot) = \overline{\psi}(h^{-1}\cdot)$. Then $\overline{\tau}$ is a faithful (normal) semifinite trace on $\overline{\mathcal{A}}$. For any positive normal faithful functional φ on $\mathcal{A}$ denote by h_φ the faithful self-adjoint operator on $\overline{H}$ such that $\overline{\rho}(\cdot) = \overline{\tau}(h_\varphi\cdot)$. The map $\varphi \mapsto h_\varphi$ can be extended to a linear bijection of $\mathcal{A}_*$ onto a linear subspace of $\widetilde{\overline{\mathcal{A}}}$, where $\widetilde{\overline{\mathcal{A}}}$ is the space of all measurable (with respect to $\overline{\tau}$) operators affiliated with $\overline{\mathcal{A}}$. We denote this image of $\mathcal{A}_*$ by $L^1(\mathcal{A})$, and endow it with a norm $\|\cdot\|_1$ such that $\|h_\varphi\|_1 = \|\varphi\|$. Now for $p \in [1, \infty[$, $L^p(\mathcal{A}) = \{k = u|k|: u \in \pi(\mathcal{A}), |k|^p \in L^1(\mathcal{A})\}$ with $\|k\|_p = \||k|^p\|_1^{1/p}$. For $p = \infty$, $L^\infty(\mathcal{A}) = \pi(\mathcal{A})$, with $\|\pi(a)\|_\infty = \|a\|$. It turns out that $L^p(\mathcal{A})$ endowed with the norms are Banach spaces sharing all the usual properties of classical L^p spaces such as the Hölder inequality, duality properties, etc.

Lemma 1. Let $\mathcal{B}$ be a von Neumann subalgebra of $\mathcal{A}$. Assume that $\psi|\mathcal{B}$ is semifinite and that $\sigma_t^{\psi|\mathcal{B}} = \sigma_t^\psi|\mathcal{B}$ for each $t \in \mathbb{R}$. Then $\overline{\mathcal{B}}$ can be canonically embedded into $\overline{\mathcal{A}}$ and for each $p \in [1, \infty]$ the space $L^p(\mathcal{B})$ can be canonically embedded into $L^p(\mathcal{A})$ so that for all $k \in L^p(\mathcal{B})$, $\|k\|_p^{\mathcal{B}} = \|k\|_p^{\mathcal{A}}$.

Proof. Note that both $\overline{\mathcal{B}}$ and $\overline{\mathcal{A}}$ act on the same Hilbert space $\overline{H}$ and that $\pi_\mathcal{B} = \pi_\mathcal{A}|\mathcal{B}$, $\lambda_\mathcal{B} = \lambda_\mathcal{A}$, $h_\mathcal{B} = h_\mathcal{A}$ and $\overline{\tau}_\mathcal{B} = \overline{\tau}_\mathcal{A}|\overline{\mathcal{B}}$. It follows from the assumptions, by virtue of a theorem of Takesaki (1972), that there exists a norm-one projection E from $\mathcal{A}$ onto $\mathcal{B}$. It is not hard to check, using the definition of the dual weight, that for any $\varphi \in \mathcal{B}_*$,

$$\overline{\varphi} = (\varphi \circ E)^{-}|\overline{\mathcal{B}}$$

Thus $h_{\varphi\circ E}^{\mathcal{A}} = h_\varphi^{\mathcal{B}}$, and $\|h_\varphi^{\mathcal{B}}\| = \|\varphi\|_\mathcal{B} = \|\varphi \circ E\|_\mathcal{A} = \|h_{\varphi\circ E}^{\mathcal{A}}\|$, which shows that for any $k \in L^1(\mathcal{B}) \subset L^1(\mathcal{A})$, $\|k\|_p^{\mathcal{B}} = \|k\|_p^{\mathcal{A}}$. It is now obvious that, for each $p \in [1, \infty]$, $\|k\|_p^{\mathcal{B}} = \|k\|_p^{\mathcal{A}}$ for $k \in L^p(\mathcal{B}) \subset L^p(\mathcal{A})$.

2. THE FINITE DISCRETE CASE

Let $\mathcal{A}$ be a finite discrete factor, and τ the faithful (normal) tracial state on $\mathcal{A}$. For each $a \in \mathcal{A}$ and $p \in [1, \infty[$, put

$$\|a\|_p^\tau = \tau(|a|^p)^{1/p}$$

For $p = \infty$, put $\|a\|_\infty^\tau = \|a\|$. It is easy to check that for each $p \in [1, \infty]$,

$\|\cdot\|_p^\tau$ is a norm turning $\mathscr{A}$ into a Banach space which we denote by $L^p(\mathscr{A}, \tau)$. Moreover, the Hölder inequality

$$\|ab\|_r^\tau \leq \|a\|_p^\tau \|b\|_q^\tau$$

holds for all $a, b \in \mathscr{A}$, with $p, q, r \in [1, \infty]$ such that $1/p + 1/q = 1/r$. Finally, for each $a \in \mathscr{A}$ and $p \in [1, \infty[$,

$$\|a\|_p^\tau = \sup_{\|b\|_q^\tau \leq 1} |\tau(ab)| \qquad \text{where} \quad q \in [1, \infty] \text{ is such that } 1/p + 1/q = 1$$

Let now φ be an arbitrary faithful (normal) state on $\mathscr{A}$. There exists a unique $h \in \mathscr{A}$ such that

$$\varphi(a) = \tau(ha) \qquad \text{for all} \quad a \in \mathscr{A}$$

Moreover, h is positive and invertible, and $\tau(h) = 1$.

For all $a \in \mathscr{A}$ and $p \in [1, \infty[$, put

$$\|a\|_p = \tau(|h^{1/2p} ah^{1/2p}|^p)^{1/p}$$

For $p = \infty$, let $\|a\|_\infty = \|a\|$. We also define the bilinear form

$$\langle a, b \rangle = \tau(h^{1/2} ah^{1/2} b) \qquad \text{for all} \quad a, b \in \mathscr{A}$$

Lemma 1. For all $p \in [1, \infty]$ we have that:

(i) $\|\cdot\|_p$ is a norm on $\mathscr{A}$

(ii) $|\langle a,b \rangle| \leq \|a\|_p \|b\|_q$, where $1/p + 1/q = 1$ and $a, b \in \mathscr{A}$.

(iii) $\|a\|_p = \sup_{\|b\|_q \leq 1} |\langle a, b \rangle|$ for all $a \in \mathscr{A}$, where $q \in [1, \infty]$ is such that $1/p + 1/q = 1$.

Proof. (i) Note that $\|a\|_p = \|h^{1/2p} ah^{1/2p}\|_p^r$. If $\|a\|_p = 0$, then $h^{1/2p} ah^{1/2p} = 0$, so that $a = 0$. Hence $\|\cdot\|_p$ is a norm.

(ii) We have, for all $p \in [1, \infty]$ and $q \in [1, \infty]$ such that $1/p + 1/q = 1$,

$$\begin{aligned} |\langle a, b \rangle| &= |\tau(h^{1/2} ah^{1/2} b)| \\ &= |\tau(h^{1/2p} ah^{1/2p} \cdot h^{1/2q} bh^{1/2q})| \\ &\leq \|h^{1/2p} ah^{1/2p} \cdot h^{1/2q} bh^{1/2q}\|_1^\tau \\ &\leq \|h^{1/2p} ah^{1/2p}\|_p^\tau \|h^{1/2q} bh^{1/2q}\|_q^\tau \\ &= \|a\|_p \|b\|_q \end{aligned}$$

(iii) Note that $h^{1/2p} \mathscr{A} h^{1/2p} = \mathscr{A}$ for each $p \in [1, \infty]$. Thus,

$$\begin{aligned} \|a\|_p &= \|h^{1/2p} ah^{1/2p}\|_p^\tau \\ &= \sup_{\|h^{1/2q} bh^{1/2q}\|_p^\tau \leq 1} |\tau(h^{1/2p} ah^{1/2p} \cdot h^{1/2q} bh^{1/2q})| \\ &= \sup_{\|b\|_q \leq 1} |\langle a, b \rangle| \end{aligned}$$

Lemma 2. If $p,\ s \in [1, \infty]$ and $p \le s$, then $\|a\|_p \le \|a\|_s$ for all $a \in \mathcal{A}$.

Proof. By the Hölder inequality, if $p \le s < \infty$, then

$$\begin{aligned}\|a\|_p &= \|h^{1/2p} a h^{1/2p}\|_p^\tau \\ &= \|h^{1/2r}(h^{1/2s} a h^{1/2s}) h^{1/2r}\|_p^\tau \\ &\le \|h^{1/2r}\|_{2r}^\tau \, \|h^{1/2s} a h^{1/2s}\|_s^\tau \, \|h^{1/2r}\|_{2r}^\tau = \|a\|_s\end{aligned}$$

In the case $s = \infty$,

$$\|a\|_p = \sup_{\|b\|_q \le 1} |\langle a, b\rangle| \le \sup_{\|b\|_1 \le 1} (\|a\| \cdot \|b\|_1) \le \|a\|$$

by the first part of the proof.

The norms $\|\cdot\|_p$ turn $\mathcal{A}$ into a Banach space which we denote by $L^p(\mathcal{A}, \varphi)$. If $\varphi = \tau$ we are back to the old space $L^p(\mathcal{A}, \tau)$. In particular,

$$\|a\|_p^\tau \le \|a\|_s^\tau \qquad \text{for} \quad p,\ s \in [1, \infty], \quad p \le s$$

It is also true that if φ_1, φ_2 are two faithful states on $\mathcal{A}$, then $L^p(\mathcal{A}, \varphi_1)$ and L^p $(\mathcal{A}, \varphi_2)$ are isomorphic (that is, isometric) Banach spaces (since both are isomorphic to $L^p(\mathcal{A}, \tau)$).

Lemma 3. For each $p \in [1, \infty]$, the Banach space $L^p(\mathcal{A}, \varphi)$ is isomorphic to the Haagerup space L^p $(\mathcal{A})$.

Proof. We may assume that $\varphi = \tau$ and $p < \infty$. Note that, since the modular group $\{\sigma_t^\tau\}$ acts trivially on $\mathcal{A}$,

$$\overline{\mathcal{A}} := \mathcal{A} \rtimes_{\sigma^\tau} \mathbb{R} \simeq \mathcal{A} \otimes L^\infty(\mathbb{R})$$

Furthermore, the canonical trace $\bar{\tau}$ on the crossed product $\overline{\mathcal{A}}$ equals $\tau \otimes e^{-s}$ ds. The Haagerup space $L^p(\mathcal{A})$ consists of products $a \otimes \exp((\cdot)/p)$, where $a \in \mathcal{A}$. Hence it is enough to show that the mapping

$$a \mapsto a \otimes \exp((\cdot)/p)$$

is an isometry. It is clear that one needs only to consider the case $p = 1$. We must show that

$$\tau(|a|) = \bar{\tau}(\chi_{]1,\infty[}(|a| \otimes \exp(\cdot)))$$

(see Terp (1981), Chapter II, Lemma 5). We calculate

$$\begin{aligned}\bar{\tau}(\chi_{]1,\infty[}(|a| \otimes \exp(\cdot))) &= \int_{-\infty}^{+\infty} \tau(\chi_{]e^{-s},\infty[}(|a|)) e^{-s}\, ds \\ &= \int_0^\infty \tau(\chi_{]s,\infty[}(|a|))\, dt = \tau(|a|)\end{aligned}$$

which completes the proof.

3. THE UHF ALGEBRAS

Let A be an UHF C^*-algebra. We consider a fixed faithful product state φ on A. Thus, there exists a sequence (B_n), $n = 1, 2, \ldots$, of mutually commuting finite discrete subfactors of A (each containing the unit of A) such that $\bigcup_{n=1}^{\infty} B_n$ generate A as a C^*-algebra, and that

$$\varphi(b_1 b_2 \cdots b_n) = \varphi(b_1)\varphi(b_2) \cdots \varphi(b_n)$$

for all $b_j \in B_j$, $j = 1, 2, \ldots, n$. Let A_n denote the finite discrete subfactor of A generated by $\bigcup_{j=1}^{n} B_j$, and put $A_\infty = \bigcup_{k=1}^{\infty} A_n$. For each n we denote by φ_n the restriction of φ to A_n and by $\|\cdot\|_p^{(n)}$ the norm of the Banach space $L^p(A_n, \varphi_n)$. It is easy to check that for $a \in A_n$, $\|a\|_p^{(n)} = \|a\|_p^{(n+k)}$ for any positive integer k. Hence we can introduce functionals $\|\cdot\|_p$ on A_∞, by putting

$$\|a\|_p = \|a\|_p^{(n)} \qquad \text{when} \quad a \in A_n$$

Obviously, $\|\cdot\|_p$ turns A_∞ into a normed space. We denote by $L^p(A, \varphi)$ the completion of A_∞ with respect to the norm. If $a \in A_n$, there exists a sequence (a_n) of elements of A_∞ converging to a in the norm of A. Hence, (a_n) is Cauchy in the norm $\|\cdot\|_p$ for each $p \in [1, \infty]$. Denote its limit in $L^p(A, \varphi)$ by $\iota_p(a)$. The map ι_p: $A \to L^p(A, \varphi)$ is an injection which extends the natural embedding of A_∞ into this completion. Thus, for each $p \in [1, \infty]$, A can be treated as a subspace of $L^p(A, \varphi)$.

Note that up to now it has not been made clear whether or not the spaces $L^p(A, \varphi)$ depend (as Banach spaces) on the approximating sequence (A_n).

Let $(H_\varphi, \pi_\varphi, \xi_\varphi)$ be the GNS representation of the pair (A, φ). Let $\mathcal{A} = \pi_\varphi(A)''$ and $\mathcal{A}_n = \pi_\varphi(A_n)$. Put $\omega := \omega_{\xi\varphi}$ and $\omega_n := \omega_{\xi\varphi}|\mathcal{A}_n$. We can treat ω as a state on $\mathcal{A}$. By Kadison and Ringrose (1986), Theorem 13.1.13, ω is faithful on $\mathcal{A}$ and the modular group $\{\sigma_t^\omega\}_{t\in\mathbb{R}}$ of $\mathcal{A}$ leaves each $\mathcal{A}_n$ invariant. There is also a natural isometric isomorphism between the Banach spaces $L^p(A_n, \varphi_n)$ and $L^p(\pi_\varphi(A), \omega)$. Lemma 1 enables us to embed $L^p(\mathcal{A}_n)$ into $L^p(\mathcal{A})$. One can show that, for any $p \in [1, \infty]$, $\bigcup_{n=1}^{\infty} L^p(\mathcal{A}_n)$ is dense in the Banach space $L^p(\mathcal{A})$. From this it is easy to conclude that the spaces $L^p(\pi_\varphi(A), \omega)$ and $L^p(\mathcal{A})$ are isomorphic as Banach spaces. Thus, we have the following result:

Theorem. Let A be a UHF C^*-algebra, φ a faithful product state on $\mathcal{A}$, and (A_n) an approximating sequence of finite discrete subfactors of A. Let $L^p(A, \varphi)$, $p \in [1, \infty]$, be the Banach space constructed at the beginning of this section. Then $L^p(A, \varphi)$ and $L^p(\pi_\varphi(A)'')$ are isomorphic as Banach spaces. In particular, the space $L^p(A, \varphi)$ does not depend on the choice of the approximating sequence.

Full details and generalizations of this result will be published elsewhere.

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